Methods of solving first order and first degree equations

These classes are classified as:→

1. Exact equations.
2. Equation solvable by separation of variables.
3. Homogeneous equations.
4. Linear equations of first order.

Remark:→ An ordinary differential equation of first order and first degree can be written as

$$\frac{dy}{dx} = f(x,y) \quad \text{———} \quad (1)$$

or in the more symmetric form

$$M\,dx + N\,dy = 0 \quad \text{———} \quad (2)$$

Where $M = M(x,y)$ & $N = N(x,y)$ are functions of x and y.

Exact Equations:→ A differential equation (1) of first order and first degree can be expressed in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Where M & N are functions of x & y.

If the differential $Mdx + Ndy$ is immediately, that is without multiplication by an factor, expressible in the form du, where u is a function of x & y, it is said to be exact & the equation $Mdx + Ndy = 0$ is then called an exact differential equation.

Example ①:→ The equation $ydx + xdy = 0$ is an exact differential equation.

$\because$ $ydx + xdy = d(x \cdot y)$

~~Its~~ Its general solution is $xy = c$.

Example ② $2x^2y\,dy + 2y^2x\,dx = 0$ is an exact differential equation,

since $2x^2y\,dy + 2y^2x\,dx = d(x^2y^2)$

Its primitive is $x^2y^2 = c$.

THEOREM:→ The necessary and sufficient condition that

$$Mdx + Ndy = 0 \quad \text{——} \quad ①$$

be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{——} \quad ②$$

Proof:→ To prove that condition is necessary.

If the equation ① is exact, then for some function $u(x,y)$ we have

$Mdx + Ndy$ = a perfect differential of $u = du$

$$\because du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

The two expressions ~~are~~ must be identical and hence, dx, dy being independent increments,

$$\frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} \quad \& \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$$

Then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, provided that the equivalent expression $\frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial^2 u}{\partial x \partial y}$ are continuous.

The tacit assumption that M & N have continuous partial derivatives. should not be overlooked.

__Sufficient condition:→__

Suppose that the condition ② is satisfied

i,e; $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Put $$F = \int M\,dx$$

Where the integration is performed on the supposition that y is constant.

Then $$\frac{\partial F}{\partial x} = M$$

and $$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

so, $\frac{\partial}{\partial x}\left(N - \frac{\partial F}{\partial y}\right) = 0$

$\therefore N - \frac{\partial F}{\partial y}$ is constant so far as x is concerned, that is, a function of $y = \phi(y)$, say

Hence $$N = \phi(y) + \frac{\partial F}{\partial y}$$

We next take function $u(x,y)$ so that

$$u = F + \int \phi(y)\, dy$$

Then

$$\frac{\partial u}{\partial y} = N$$

Also

$$M = \frac{\partial F}{\partial x} \text{ (by our construction of F)}$$

$$= \frac{\partial u}{\partial x} \quad \left(\because u = F + \int \phi(y)\, dy\right)$$

Thus,

$$M dx + N dy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = du$$

is a perfect differential.

proved.

Example ① Is the equation

$$(x^2-y)\,dx + (y^2-x)\,dy = 0$$ an exact equation?

Solution:→

Here $M = x^2-y$, $N = y^2-x$

$$\therefore \frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x}$$

∴ The equation is exact.

Working Rule:→

When the condition of integrability is satisfied; the general solution (or the primitive) can be obtained by the following rules:

① Integrate $M\,dx$ as if y is constant.

② Integrate $N\,dy$ as if x is constant but, then delete those terms already ~~contained~~ obtained in ①

③ The sum of these integrals will give a function $u(x,y)$ such that

$$du = M\,dx + N\,dy$$

④ Once the function u is obtained, then $u = c$ is the primitive.

Example (2) Solve $(x+y)\,dy - (x-y)\,dx = 0$

Solution: → From the given equation, we have

$$(x-y)\,dx - (x+y)\,dy = 0$$

Here $M = x - y$, $N = -(x+y)$

$$\therefore \frac{\partial M}{\partial y} = -1 \quad \& \quad \frac{\partial N}{\partial x} = -1$$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ & hence the given equation is **exact**.

$$\because (x-y)\,dx - (x+y)\,dy = 0$$

$$\Rightarrow x\,dx - y\,dx - x\,dy - y\,dy = 0$$

$$\Rightarrow x\,dx - y\,dy - (x\,dy + y\,dx) = 0$$

$$\Rightarrow x \cdot dx - y \cdot dy - d(x \cdot y) = 0$$

integrating, we have

$$\frac{x^2}{2} - \frac{y^2}{2} - xy = c$$

$$\Rightarrow x^2 - y^2 - 2xy = 2c$$

Ans.

Example (3) Solve $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

Solution: →

Here $M = x^2 - 4xy - 2y^2$, $N = y^2 - 4xy - 2x^2$

$$\therefore \frac{\partial M}{\partial y} = -4x - 4y \quad \& \quad \frac{\partial N}{\partial y} = -4y - 4x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given equation is exact.

Therefor the solution of the equation is

$$\int_{y=\text{const.}} (x^2-4xy-2y^2)\,dx + \int y^2 dy = c$$

$$\Rightarrow \frac{x^3}{3} - 4y\cdot\frac{x^2}{2} - 2y^2x + \frac{y^3}{3} = c$$

$$\Rightarrow x^3 - 6x^2 y - 6y^2 x + y^3 = c$$

$$\Rightarrow x^3+y^3-6xy(x+y) = c$$

Ans.

Example ④ solve $(2x-y+1)dx + (2y-x-1)dy = 0$ — ①

Solution :→

Here, $M = 2x-y+1$; $N = 2y-x-1$

$$\therefore \frac{\partial M}{\partial y} = -1 \text{ and } \frac{\partial N}{\partial x} = -1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given differential equation is exact.

$$\because (2x-y+1)dx + (2y-x-1)dy = 0$$

$$\Rightarrow 2x\,dx - (y\,dx + x\,dy) + dx + 2y\,dy - dy = 0$$

$$\Rightarrow 2x\,dx - d(x\cdot y) + dx + 2y\,dy - dy = 0$$

Hence integrating, we have

$$2\cdot\frac{x^2}{2} - xy + x + 2\cdot\frac{y^2}{2} - y = c$$

i.e; $x^2 - xy + x + y^2 - y = c$

Which is the required solution.

Ex⑤ Solve $\frac{dy}{dx} = \frac{6x-2y-7}{2x+3y-6}$ — ①

Sol^n:→ $\therefore (2x+3y-6)\,dy = (6x-2y-7)\,dx$

$\Rightarrow (2x+3y-6)\,dy - (6x-2y-7)\,dx = 0$

$\Rightarrow (6x-2y-7)\,dx - (2x+3y-6)\,dy = 0$ — ②

Here, we have $M = 6x-2y-7$; $N = -(2x+3y-6)$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -2.$

$\therefore$ The given differential equation is exact.

Now from ②, we have

$6x\,dx - 2y\,dx - 7dx - 2x\,dy - 3y\,dy + 6dy = 0$

$\Rightarrow 6x\,dx - 2(y\,dx + x\,dy) - 7\,dx - 3y\,dy + 6\,dy = 0$

$\Rightarrow 6x\,dx - 2\,d(x\cdot y) - 7\,dx - 3y\,dy + 6\,dy = 0$

Integrating, we have

$6\cdot\frac{x^2}{2} - 2xy - 7x - 3\cdot\frac{y^2}{2} + 6y = c$

$\Rightarrow 3x^2 - 2xy - 7x - \frac{3}{2}y^2 + 6y = c$

Ans.